McGill University March 2012
Faculty of Engineering Final examination

Differential Equations for Engineers

Math 263

April 23, 2012

(Time: 2:00pm — 5:00pm)

Examiner: Prof. J.J. Xu Associate Examiner: Matthew Roberts

Student name (last, first)	**Student number (McGill ID)**

INSTRUCTIONS

1. This is a closed book exam. No notes allowed.
2. Faculty Standard Calculators and Translation Dictionaries are permitted.
3. Before you begin, please take a couple of minutes to scan the problems. (Please inform the invigilator if the booklet is defective.)
4. You are expected to show all your work. All solutions are to be written on the page where the problem is printed. You may continue your solutions on the facing page. If that space is exhausted you may continue on the blank pages at the end, clearly indicating any continuation on the page where the question is printed.

This exam comprises the cover page, eight pages of questions numbered 2 to 9, five blank pages numbered 11 to 15, and the Table of Laplace Transforms.

Problem	1	2	3	4	5	6	7	8	Total
Mark									
Out of	11	11	10	14	14	15	14	11	100

1. (**9 points**) Consider the following ODE:

$$y + (2xy - \mathrm{e}^{-y})y' = 0.$$

(a) Determine whether or not it is an exact equation. If it is not, then find an integrating factor for the equation.

(b) Find the general solution of the equation.

Solution:

(a) It is not exact equation; it has the integrating factor

$$\mu(y) = \frac{\mathrm{e}^{2y}}{y}.$$

(b) The general solution is

$$\begin{cases} x\mathrm{e}^{2y} - \ln|y| = C, \\ y = 0. \end{cases}$$

2. (**9 points**) Consider the differential equation

$$(x-y)y' - (3x-y) = 0.$$

(a) Find the general solution of the above equation.

(b) Solve the IVP of the above equation with the initial condition $y(1) = y_0 > 0$.

Solution: Let $u = y/x$.

$$xu' = \frac{3-u}{1-u} - u = \frac{3-2u+u^2}{1-u} = \frac{(u-1)^2+2}{1-u}$$

So that

$$\int \frac{u-1}{(u-1)^2+2}\mathrm{d}u = -\int \frac{1}{x}\mathrm{d}x.$$

we have the solution:

$$\frac{1}{2}\ln|(u-1)^2+2| = -\ln|x| + C$$

3. (**10 points**)

(a) Write down an annihilator $Q(D)$ for $e^x \sin\sqrt{3}x$.

(b) Find the general solution to

$$y'' + 6y' + 20y = 8e^x \sin\sqrt{3}x.$$

Solution:

(a) $Q(D) = (D-1)^2 + 3 = D^2 - 2D + 4 = (D - 1 - \sqrt{3}i)(D - 1 + \sqrt{3}i)$.

(b) The characteristic polynomial for the homogeneous equation is $P(r) = r^2 + 6r + 20$ which has roots $-3 \pm \sqrt{11}i$, which gives the solution for the homogeneous part to be $Ae^{-3x}\sin\sqrt{11}x + Be^{-3x}\cos\sqrt{11}x$. Now solving $Q(D)(y_p) = 0$ gives

$$\begin{aligned} y_p &= ae^x \sin\sqrt{3}x + be^x \cos\sqrt{3}x \\ y_p' &= (a - \sqrt{3}b)e^x \sin\sqrt{3}x + (\sqrt{3}a + b)e^x \cos\sqrt{3}x \\ y_p'' &= (-2a - 2\sqrt{3}b)e^x \sin\sqrt{3}x + (2\sqrt{3}a - 2b)e^x \cos\sqrt{3}x. \end{aligned}$$

Thus

$$y_p'' + 6y_p' + 20y_p = (24a - 8\sqrt{3}b)e^x \sin\sqrt{3}x + (8\sqrt{3}a + 24b)e^x \cos\sqrt{3}x$$

so we need

$$24a - 8\sqrt{3}b = 8 \quad \text{and} \quad 8\sqrt{3}a + 24b = 0$$

which gives $a = 1/4$, $b = \sqrt{3}/12$. Thus the general solution is

$$y = Ae^{-3x}\sin\sqrt{11}x + Be^{-3x}\cos\sqrt{11}x + \frac{1}{4}e^x \sin\sqrt{3}x - \frac{1}{4\sqrt{3}}e^x \cos\sqrt{3}x.$$

4.(**14 points**) Note that $y = x + 1$ is a solution to the homogeneous equation

$$\frac{(x+1)^2}{x(x+2)}y'' + (x+1)y' - y = 0, \quad x > 0.$$

By the reduction of order method, or otherwise, find the general solution to the inhomogeneous equation:

$$\frac{(x+1)^2}{x(x+2)}y'' + (x+1)y' - y = \frac{(x+1)^3}{x(x+2)}, \quad x > 0.$$

Solution: For the reduction of order method we try $y = (x+1)C(x)$. This gives

$$\begin{aligned} y &= (x+1)C \\ y' &= C + (x+1)C' \\ y'' &= 2C' + (x+1)C'' \end{aligned}$$

and substituting back into the (inhomogeneous) equation we get

$$\frac{(x+1)^3}{x(x+2)}C'' + (x+1)^2\left(\frac{2}{x(x+2)} + 1\right)C' = \frac{(x+1)^3}{x(x+2)}$$

so

$$C'' + \frac{1}{x+1}(2 + x(x+2))C' = 1$$

which simplifies to

$$C'' + \left(x + 1 + \frac{1}{x+1}\right)C' = 1.$$

We now use the integrating factor $e^{(x+1)^2/2+\ln(x+1)} = (x+1)e^{(x+1)^2/2}$: our equation becomes

$$\frac{d}{dx}((x+1)e^{(x+1)^2/2}C') = (x+1)e^{(x+1)^2/2}$$

so

$$(x+1)e^{(x+1)^2/2}C' = e^{(x+1)^2/2} + c_1.$$

Then $C' = (x+1)^{-1} + c_1(x+1)^{-1}e^{-(x+1)^2/2}$, so $C = \ln(x+1) + c_1\int(x+1)^{-1}e^{-(x+1)^2/2}dx + c_2$, which gives the solution

$$y = c_0(x+1) + c_1\int(x+1)^{-1}e^{-(x+1)^2/2}dx + (x+1)\ln(x+1).$$

5. (**10 points**) Recall that $\cosh x = (e^x + e^{-x})/2$. Compute the Laplace transform of the following functions $f(t)$:

(a)

$$f(t) = t \cosh at, \quad (a > 0);$$

(b)

$$f(t) = \begin{cases} e^t, & (0 \le t < \pi/4); \\ e^t + \cos(t - \pi/4), & (t \ge \pi/4). \end{cases}$$

Solution:

$$f(t) = e^t + g(t)$$

where

$$g(t) = \begin{cases} 0, & (0 \le t < \pi/4); \\ \cos(t - \pi/4), & (t \ge \pi/4). \end{cases}$$

So,

$$g(t) = u_{\pi/4}(t) \cos(t - \pi/4).$$

$$\mathcal{L}[f(t)] = \mathcal{L}[e^t] + \mathcal{L}[u_{\pi/4}(t)\cos(t - \pi/4)]$$
$$= \tfrac{1}{s-1} + e^{-\pi s/4}\mathcal{L}[\cos t] = \tfrac{1}{s-1} + e^{-\pi s/4}\tfrac{s}{s^2+1}.$$

6. (**14 points**) Solve the following initial value problem:

$$y^{(4)} + 5y'' + 4y = 1 - u_\pi(t), \quad y(0) = y'(0) = y''(0) = y'''(0) = 0$$

where

$$u_\pi(t) = \begin{cases} 0 & t < \pi, \\ 1 & t \geq \pi. \end{cases}$$

Solution:

$$y(t) = h(t) - u_\pi(t)h(t - \pi), \quad h(t) = (3 - 4\cos t + \cos 2t)/12.$$

7.(**12 points**) Consider the ODE

$$y'' - \frac{1}{2x}y' - \frac{3}{x}y = 0, \quad x > 0.$$

One of the non-zero solutions to this equation can be written in the form

$$y_1 = x^{3/2} \sum_{n=0}^{\infty} \frac{(n+1)c^n}{(pn+q)!} x^n,$$

for some positive integers c, p and q. Determine c, p and q.

Solution: There are a few ways of doing this, but here is my favourite (which uses theory from the course). The indicial equation is

$$F(r) = r(r-1) - r/2 = r^2 - 3r/2$$

which has roots 3/2 and 0. Clearly we are interested in the root 3/2. This gives us the recurrence

$$a_n = \frac{3a_{n-1}}{(n+3/2)^2 - 3(n+3/2)/2} = \frac{3a_{n-1}}{n(n+3/2)} = \frac{12a_{n-1}(n+1)}{n(2n+3)(2n+2)}.$$

By repeated application of this recurrence (or induction) we get

$$a_n = \frac{12^n a_0(n+1)}{(2n+3)!/6}.$$

To complete the question we may take $a_0 = 1/6$, so $A = 12$ and $P(n) = 2n+3$.

Students can be given marks for recognising that this is Frobenius territory, for finding the roots of the indicial equation, and then for getting the correct recurrence relation. If, on the other hand, they try to do things by hand by direct substitution into the equation, they are likely to either make a complete mess of things (very few marks) or get the correct solution (full marks).

8.(**12 points**) Consider the ODE

$$y'' + (x-1)y = \ln x, \quad 0 < x < 2, \quad y(1) = y'(1) = 0.$$

By substituting

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n,$$

find a recurrence relation for a_{n+3} in terms of a_n. Calculate a_3, a_4 and a_5.

Solution:

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n (x-1)^n \\
y' &= \sum_{n=0}^{\infty} n a_n (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n \\
y'' &= \sum_{n=0}^{\infty} n(n+1) a_{n+1} (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-1)^n
\end{aligned}$$

and

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n.$$

Thus we need

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x-1)^n + \sum_{n=1}^{\infty} a_{n-1} (x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n.$$

This gives $a_2 = 0$ and

$$(n+1)(n+2) a_{n+2} + a_{n-1} = \frac{(-1)^{n+1}}{n}$$

which translates to

$$(n+2)(n+3) a_{n+3} + a_n = \frac{(-1)^n}{n+1}.$$

Now, from the initial conditions, we must have $a_0 = a_1 = 0$. So $a_3 = 1/6$, $a_4 = -1/24$ and $a_5 = 1/60$.

9.(**10 points**) Consider the IVP specified by the following system of equations:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A\mathbf{x}, \quad \mathbf{x}(0) = (1,1,0)^T,$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of the matrix A.

(b) Use the method of eigenvalues and eigenvectors to find the general solution of the system.

(c) Give the solution for the IVP.

Solution: The general solution:

$$\mathbf{x} = c_1\mathrm{e}^{-t}[-3,4,2]^T + c_2\mathrm{e}^{2t}[0,1,-1]^T + c_3\mathrm{e}^{2t}\Big\{[0,1,-1]^T t + [1,0,1]^T\Big\}$$

with IC: $\mathbf{x}(0) = [1,1,0]^T$, we have

$$\begin{aligned} c_1(-3) + c_2(0) + c_3(1) = 1; \\ c_1(4) + c_2(1) + c_3(0) = 1; \\ c_1(2) + c_2(-1) + c_3(1) = 0. \end{aligned}$$

We derive

$$c_1 = 0, c_2 = 1, c_3 = 1.$$

$$\mathbf{x} = \mathrm{e}^{2t}[0,1,-1]^T + \mathrm{e}^{2t}\Big\{[0,1,-1]^T t + [1,0,1]^T\Big\} = \mathrm{e}^{2t}[1,1,0]^T + t\mathrm{e}^{2t}[0,1,-1]^T.$$

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